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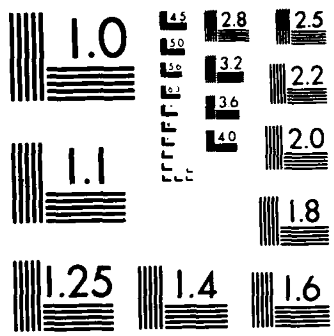

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Sound Waves in a Medium Containing Rigid Spheres

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Sound waves in a medium containing rigid spheres*

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Abstract. The effective speed of sound is calculated for a medium containing immovable rigid spheres arranged in a simple cubic lattice. Long waves propagating along a lattice axis are treated. The wave equation for the pressure is reduced to an ordinary differential equation to which Floquet theory is applied. Both perturbation and numerical methods are use to find the effective speed as a function of frequency, and to locate the pass and stop bands.

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1. Introduction

The effective speed of sound in a medium containing particles is different from that in the ambient medium. We calculate it for the idealized case of immovable rigid spheres with centers arranged in a simple cubic lattice. We assume that the wavelength in the ambient medium is large compared to the distance between particles. Then the wave equation for the pressure can be reduced to an ordinary differential equation with periodic coefficients. Floquet theory shows that there are pass bands and stop bands along the frequency axis. We determine them, and the sound speed in the pass bands, by both analytical and numerical means.

2. Formulation

Let the x-axis be a lattice axis with L the spacing between sphere centers along it, and let R be the radius of each sphere. A wave of frequency ω propagates along this axis. By the periodicity of the lattice, the wave is symmetric about the planes $y = \pm L/2$ and $z = \pm L/2$. Therefore it suffices to determine the wave within the region bounded by these four planes, which may be thought of as a rigid-walled waveguide of square cross section with spheres placed periodically along its axis. When $\omega L/c$ is small, where c is the sound speed in the ambient medium, the wavelength in this medium is large compared to L. Then the pressure p is practically constant over the cross-section of the waveguide, so we write it as $p(x)$.

Under these conditions, $p(x)$ satisfies the long wave equation [L1]

$$p_{xx} + \frac{S_x}{S} p_x + k^2 p = 0, \qquad k = \omega/c. \tag{2.1}$$

Here $S(x)$ is the unobstructed cross-sectional area of the guide, given by

$$S(x) = L^2 - \pi(R^2 - x^2), \qquad |x| \leq R \tag{2.2}$$

$$S(x) = L^2, \qquad R \leq |x| \leq L/2$$

$$S(x+L) = S(x).$$

In view of (2.2), the ordinary differential equation (2.1) has a periodic coefficient. By Floquet's theorem it has a solution which satisfies the condition

$$p(x+L) = e^{iKL} p(x) \tag{2.3}$$

for some constant $K(k)$ which depends upon k. The complex conjugate p^* of p is another solution which satisfies (2.1) with iK replaced by $-iK^*$. When K is real, (2.3) represents a wave propagating in the direction of increasing x with wavenumber K. Therefore its phase velocity is $C = \omega/K$ and its group velocity is $C_g = (dK/d\omega)^{-1}$. In terms of k these relations become

$$\frac{C}{c} = \frac{k}{K}, \qquad \frac{C_g}{c} = \frac{dk}{dK} = \left(\frac{dK}{dk}\right)^{-1}. \tag{2.4}$$

When K is not real p increases or decreases exponentially with x and the wave is called evanescent or nonpropagating. In this case we say that ω and k lie in a stop band, while when K is real they lie in a pass band.

Thus our task is to find a solution p of (2.1) and a constant $K(k)$ such that (2.3) holds. Instead we shall specify K and seek $k(K)$ and p satisfying (2.1) and (2.3) because this is a standard eigenvalue problem with eigenvalue k^2.

It will be convenient to introduce dimensionless variables with L as the unit of length. Thus we set $x' = x/L$, $R' = R/L$, $k' = kL$, $K' = KL$ and $S' = S/L^2$. From now on we will use these variables, omitting the prime.

3. Perturbation expansion

The coefficient of p_x in (2.1) is $S_x/S = 2\pi x/S$ for $|x| < R$ and zero for $R < |x| < 1/2$. Its maximum absolute value is $2\pi R$ when $R \le (2\pi)^{-\frac{1}{2}}$, so it is small when R is small. Therefore we shall solve the problem by a perturbation expansion, treating S_x/S as small. The results will be valid for R small, although we shall see that they are useful even for $R = 1/2$, when adjacent spheres touch one another.

To proceed we write

$$p(x) = p_0(x) + p_1(x) + p_2(x) + \dots \tag{3.1}$$

$$k^2 = k_0^2 + (k^2)_1 + (k^2)_2 + \dots .$$

A term with subscript j is of jth order in S_x/S. Now we substitute (3.1) into (2.1) and equate terms of each order to obtain

$$p_0'' + k_0^2 p_0 = 0, \tag{3.2}$$

$$p_1'' + k_0^2 p_1 = -\frac{S'}{S} p_0' - (k^2)_1 p_0, \tag{3.3}$$

$$p_2'' + k_0^2 p_2 = -\frac{S'}{S} p_1' - (k^2)_2 p_0 - 2(k^2)_1 p_1. \tag{3.4}$$

We also find that each p_j satisfies (2.3).

By writing $p_0 = e^{ik_0 x}$ and then using (2.3) to determine k_0 we find an infinite number of solutions:

$$p_0 = e^{ik_0 x}, \quad k_0 = K + 2\pi m, \quad m = 0, \pm 1, \dots . \tag{3.5}$$

When K is real then k_0 is real. Before solving (3.3) we multiply it by $p_0^* = e^{-ik_0 x}$ and integrate the resulting equation from $-1/2$ to $1/2$. After integrating by parts and using (2.3) we find that the integral of the left side vanishes. Thus the integral of p_0^* times the right side vanishes, so we can solve it for $(k^2)_1$ and obtain

$$(k^2)_1 = -ik_0 \int_{-1/2}^{1/2} \frac{S'(x)}{S(x)} dx = 0. \tag{3.6}$$

Next we proceed in the same way with (3.4) and again the integral of p_0^* times the left side vanishes. Then the integral of p_0^* times the right side yields

$$(k^2)_2 = -\int_{-1/2}^{1/2} e^{-ik_0 x} \frac{S'(x)}{S(x)} p_1'(x) dx. \tag{3.7}$$

Now we solve (3.3) by means of a modified Green's function $G(x, y)$, to be determined below, and we get

$$p_1(x) = -ik_0 \int_{-1/2}^{1/2} G(x, y) \frac{S'(y)}{S(y)} e^{ik_0 y} dy. \tag{3.8}$$

Then we use (3.8) in (3.7) and find

$$(k^2)_2 = ik_o \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-ik_0 x}\frac{S'(x)}{S(x)}G_x(x,y)\frac{S'(y)}{S(y)}e^{ik_0 y}dy\,dx. \qquad 3.9$$

The results (3.5), (3.6) and (3.9) yield the first three terms in the expansion of k^2, while (3.5) and (3.8) yield two terms in the expansion of $p(x)$.

To complete the calculation we must determine G which is defined by

$$G_{xx} + k_0^2 G = \delta(x-y) - e^{ik_0(x-y)}, \qquad -1/2 \le x \le 1/2 \qquad (3.10)$$

$$G(1/2,y) = e^{iK}G(-1/2,y),$$

$$G_x(1/2,y) = e^{iK}G_x(-1/2,y).$$

The solution of (3.10) is not unique when k_0 satisfies (3.5). One solution is

$$G(x,y) = \frac{e^{ik_0(x-y)}}{2ik_0} + e^{ik_0(y-x)}\frac{e^{2iK}}{2ik_0(1-e^{2iK})} - \frac{xe^{ik_0(x-y)}}{2ik_0}, \qquad y \le x \le 1/2,$$
$$\frac{e^{ik_0(y-x)}}{2ik_0(1-e^{2iK})} - \frac{xe^{ik_0(x-y)}}{2ik_0}, \qquad -1/2 \le x \le y. \qquad (3.11)$$

By using (3.11) for G in (3.8) and (3.9) we can calculate $p_1(x)$ and $(k^2)_2$.

Instead of (3.11) we shall represent G, in terms of the eigenfunctions (3.5):

$$G_n(x,y) = \sum_{m\neq n} e^{i(K+2\pi m)(x-y)}\left[\left(K+2\pi n\right)^2 - \left(K+2\pi m\right)^2\right]^{-1}. \qquad (3.12)$$

Here we have set $k_0 = K + 2\pi n$ and denoted G by G_n. We shall also denote p_1 by p_{n1} and $(k^2)_2$ by $(k^2)_{n2}$. Then (3.8) yields

$$p_{n1}(x) = -i\sum_{m\neq n} e^{i(K+2\pi m)x}V_{nm}\left[\left(K+2\pi n\right)^2 - \left(K+2\pi m\right)^2\right]^{-1}. \qquad (3.13)$$

The matrix elements V_{nm} are defined by

$$V_{nm} = \int_{-1/2}^{1/2}\frac{S'(y)}{S(y)}e^{i2\pi(n-m)y}dy. \qquad (3.14)$$

Similarly (3.9) yields

$$(k^2)_{n2} = -(K + 2\pi n) \sum_{m \neq n} (K + 2\pi m)|V_{nm}|^2[(K + 2\pi n)^2 - (K + 2\pi m)^2]^{-1}. \tag{3.15}$$

Let us now set $n = 0$ and simplify (3.15) in the long wavelength case when K is small. We obtain

$$(k^2)_{02} = \frac{K^2}{2\pi} \sum_{m \neq 0} |V_{0m}|^2 \left[m^{-1} - \frac{K}{2\pi m^2} + O(K^2)\right]. \tag{3.16}$$

The first sum vanishes, so we get

$$(k^2)_{02} = -\frac{K^2}{2\pi^2} \sum_{m \geq 1} |V_{0m}|^2 m^{-2} + O(K^2). \tag{3.17}$$

From (3.5) see that $k_{00} = K$, while (3.6) yields $(k^2)_{01} = 0$. We now use these values and (3.17) in (3.1) to get k^2. Then we take the square root, divide by K and use (2.4) to write

$$\frac{C(R)}{C} = \frac{k}{K} = \left[1 - \frac{1}{2\pi^2} \sum_{m \geq 1} |V_{0mm}|^2 m^{-2} + \cdots\right]^{-\frac{1}{2}}. \tag{3.18}$$

The terms retained on the right side of (3.18) are independent of K, so to this order $C_g = C$.

We have evaluated V_{0m} by numerical integration with $1 \leq m \leq 12$, for five values of R. The results for $C(R)/c$ from (3.18) are

$$\frac{C(0.5)}{c} = 0.87, \quad \frac{C(0.4)}{c} = 0.966, \quad \frac{C(0.3)}{c} = 0.991, \quad \frac{C(0.2)}{c} = 0.999, \quad \frac{C(0.1)}{c} = 1.000. \tag{3.19}$$

These results can be compared with those obtained by direct numerical solution of (2.1)–(2.3), which are listed in Table 1. We see that for small K the agreement is good for all values of R, while for small R the agreement is good for all values of K. In fact since $V_{mn} = O(R^3)$, (3.18) shows that $C(R)/c = 1 - O(R^6)$, so $C(R)/c$ differs little from unity for R small. This is in agreement with the values in (3.19) and in Table 1.

The matrix elements V_{mn} given by (3.14) can be evaluated approximately when R is small, which may be useful. First we integrate by parts in (3.14) to get

$$V_{mn} = 2\pi i(m - n) \int_{-R}^{R} (\log S) e^{2\pi i(m-n)x} dx. \tag{3.20}$$

In (3.20) we write $\log S = -f - f^2/2 - f^3/3 + \ldots$ where $f(x) = \pi(R^2 - x^2)$, and obtain

$$V_{mn} = -2\pi i(m-n)\int_{-R}^{R} f(x)e^{2\pi i(m-n)x}dx + O(R^5). \tag{3.21}$$

Integration yields, with $a_{mn} = 2\pi(m-n)$,

$$V_{mn} = \frac{4\pi}{a_{mn}}\left[\frac{1}{a_{mn}}\sin(a_{mn}R) - R\cos(a_{mn}R)\right] + O(R^5). \tag{3.22}$$

From (3.22) we see that when $|a_{mn}R| \ll 1$ then $V_{mn} = a_{mn}v + O(R^5)$ where $v = \frac{4}{3}\pi R^3$ is the volume of a single sphere.

The results (3.13) and (3.15) are not valid when any denominator in them vanishes. This occurs when $K = n\pi$ for some integer n. Then $k_0 = n\pi$ and the general solution of (3.2) satisfying (2.3) is $p_0 = c_1 e^{ik_0x} + c_2 e^{-ik_0x}$ where c_1 and c_2 are constants. To find $(k^2)_1$ we multiply (3.3) by $e^{\pm ik_0x}$ and integrate from $-1/2$ to $1/2$. The integral of the left side vanishes yielding two equations:

$$\begin{aligned} (k^2)_1 c_1 - ik_0 V_{n0}^* c_2 &= 0, \\ ik_0 V_{n0} c_1 + (k^2)_1 c_2 &= 0. \end{aligned} \tag{3.23}$$

For c_1 and c_2 to be non-zero, the determinant of their coefficients must vanish, from which we get the two solutions

$$(k^2)_1^{\pm} = \pm k_0 |V_{n0}|. \tag{3.24}$$

Since $S(x)$ is symmetric, V_{n0} is imaginary. Then for $(k^2)_1 = (k^2)_1^+$ (3.23) yields $c_1/c_2 = 1$ and hence $p_0 = p_0^+ = e^{ik_0x}(1 + e^{i2\pi nx})$. Similarly for $(k^2)_1 = (k^2)_1^-$ (3.23) gives $c_2/c_1 = -1$ and $p_0 = p_0^- e^{ik_0x}(1 - e^{-i2\pi nx})$.

Next, we calculate p_1. We write p_1 as $e^{ik_0x}v_1(x)$ where $v_1(x)$ has period 1 and the Fourier expansion $\sum_m a_m e^{i2\pi mx}$ where the a_m are constants. To find a_l we substitute $p_1 = e^{ik_0x}(\sum_m a_m e^{i2\pi mx})$ into (3.3), multiply by $e^{-i(k_0+2\pi l)}$ and integrate the resulting equation from $-1/2$ to $1/2$. In this way we obtain

$$p_1^{\pm}(x) = -ik_0 e^{ik_0x} \sum_{m\neq 0,-n} \frac{(V_{m0}^* \mp V_{n+m,0}^*)}{[k_0^2 - (2\pi m + k_0)^2]} e^{i2\pi mx}. \tag{3.25}$$

To calculate $(k^2)_2$ we substitute (3.25) into (3.4), multiply by e^{-ik_0x} and integrate from $-L/2$ to $L/2$. The integral of the left side vanishes and we obtain for $K_n = \pi n/L$,

$$(k^2)_2^{\pm} = \frac{n}{4}\left(\sum_{m\neq 0,-n} \frac{(n+2m)}{(n+m)m}|V_{m0}|^2 \pm \sum_{m\neq 0,-n} \frac{(n+2m)}{(m+n)m} V_{m0}V^*_{n+m,0}\right) \tag{3.26}$$

Since $V_{m0} = i\int_{-1/2}^{1/2} \frac{S'}{S}\sin(2\pi mx)dx$ we have $V^*_{m0} = V_{-m0}$. Also, for $m = -(l+n)$, $\frac{(n+2m)}{(n+m)m}V_{m0}V^*_{n+m,0} = -\frac{(n+2l)}{(n+l)l}V_{l0}V_{n+l,0}$. Hence the second term in (3.26) vanishes and (3.26) becomes

$$(k^2)_2^{\pm} = \frac{n}{4}\sum_{m\neq 0,-n}\frac{(n+2m)}{(n+m)m}|V_{m0}|^2. \tag{3.27}$$

We now use the results (3.24) for $(k^2)_1^{\pm}$ and (3.27) for $(k^2)_2^{\pm}$ in (3.1) for k^2 with $k_0 = K = n\pi$. Denoting k by k_n we get

$$(k_n^2)^{\pm} = (n\pi)^2 \pm n\pi|V_{n0}| + \frac{n}{4}\sum_{m\neq 0,-n}\frac{n+2m}{(n+m)n}|V_{m0}|^2 + \cdots. \tag{3.28}$$

The two positive values k_n^+ and k_n^- determined by (3.28) are the two boundaries of the n-th stop band corresponding to $K = n\pi$. the width of this band is $\Delta k_n = k_n^+ - k_n^- = |V_{n0}| + \cdots$. This width is $2\pi nv + \cdots$ when $2\pi nR$ is small. The values of $\pi^{-1}k_n^{\pm}$ and of $\pi^{-1}\Delta k_n$ are shown in Table 3, based upon (3.28). The V_{m0} were calculated by numerical integration for $m \leq 12$ for five values of R for the first three bands, $n = 1, 2, 3$. The results agree well with those in Table 2, obtained by numerical solution of the eigenvalue problem (4.1) with $N + 1 = 100$.

4. Numerical solutions

To solve the eigenvalue problem (2.1–2.3) numerically we discretize (2.1) on $\{0 \leq x \leq L\}$ using a uniform grid G^h, where h is the meshsize. The resulting second order accurate conservative discretization scheme for (2.1–2.3) is

$$\frac{S_{i+\frac{1}{2}}}{S_i}(P^h_{i+1} - P^h_i) - \frac{S_{i-\frac{1}{2}}}{S_i}(P^h_i - P^h_{i-1}) + (k^h h)^2 P_i = 0 \tag{4.1}$$

$$P_{-1}^h = e^{-ikhL} P_{N-1}^h, \quad P_N^h = e^{iKL} P_0^h, \quad S_{-\frac{1}{2}} = S_{\frac{1}{2}}.$$

Here P_j^h is the solution at the gridpoint $x_j = jh$, $S_\alpha = S(\alpha h)$ and $N + 1$ is the number of gridpoints in G^h. For any given K there are $N - 1$ solutions k^h of the eigenvalue problem (4.1). In our calculations we took $L = 1$ and $N - 1 = 100$.

In Table 1, the numerical values of $k^h(R)/K$ are given for the first two bands, corresponding to K/pi in the ranges 0 to 1 and 1 to 2 respectively. We have seen that the perturbation results (3.19) are in good agreement, with them for K small and also for R small.

In Table 2, $\pi^{-1}[k_n^h(R)]^{\pm}$ is given for $n = 1, 2, 3$ where $[k_n^h]^{\pm}$ are the two values of k^h at the ends of the n-th stop band. For comparison, Table 3 shows the corresponding values of k given by (3.28), obtained by second order perturbation analysis, which agree well with the values in Table 2.

5. Conclusion

We have presented two methods for calculating the pressure $p(x)$, the phase velocity C and the propagation constant K for sound waves in a medium containing fixed rigid spheres arranged in a simple cubic lattice. One is a perturbation expansion in R/L, where R is the radius of a sphere and L is the distance between centers. The other is a direct numerical method. The results of the two methods agree well both when R/L is small and also when KL is small.

The results show that there are pass bands and stop bands along the axis of $kL = \omega L/c$. The boundaries of the first three stop bands are given in Tables 2 and 3 for five values of R/L. The values of C/c in the first two pass bands are given in Table 1 for five values of R/L and ten values of KL/π. They show that $C/c \leq 1$ in the first pass band, and that C/c decreases as R/L increases and as KL increases. In the second band $C/c \geq 1$ except near the upper edge of the band. Within this band C/c still decreases as KL increases, but it increases as R/L increases.

K/π	k^h/K				
	$R = 0.1$	$R = 0.2$	$R = 0.3$	$R = 0.4$	$R = 0.5$
0.1	1.000	0.999	0.991	0.967	0.889
0.2	1.000	0.999	0.991	0.966	0.885
0.3	1.000	0.999	0.991	0.964	0.880
0.4	1.000	0.999	0.990	0.961	0.873
0.5	1.000	0.998	0.989	0.957	0.862
0.6	1.000	0.998	0.987	0.951	0.847
0.7	1.000	0.998	0.984	0.940	0.824
0.8	1.000	0.997	0.978	0.922	0.793
0.9	1.000	0.995	0.964	0.889	0.747
1.0	0.996	0.970	0.912	0.825	0.686
1.0	1.004	1.030	1.091	1.191	1.371
1.1	1.000	1.004	1.032	1.106	1.259
1.2	1.000	1.001	1.017	1.066	1.187
1.3	1.000	1.001	1.010	1.045	1.139
1.4	1.000	1.000	1.007	1.033	1.108
1.5	1.000	1.000	1.005	1.026	1.086
1.6	1.000	1.000	1.004	1.021	1.070
1.7	1.000	1.000	1.002	1.017	1.052
1.8	1.000	0.999	1.001	1.014	1.048
1.9	0.996	0.982	0.998	1.012	1.039
2.0	0.996	0.982	0.982	1.005	1.018

TABLE 1.

Values of k^h/K obtained by numerical solution of the eigenvalue problem (4.1) with $N + 1 = 100$ for various values of K and R.

K/π		$R = 0.1$	$R = 0.2$	$R = 0.3$	$R = 0.4$	$R = 0.5$
1	$\pi^{-1}(k^h)^-$	0.9959	0.9698	0.9117	0.827	0.697
	$\pi^{-1}(k^h)^+$	1.004	1.0298	1.0907	1.191	1.371
	$\pi^{-1}\Delta k^h$	0.0082	0.0601	0.179	0.366	0.685
2	$\pi^{-1}(k^h)^-$	1.9924	1.9648	1.9634	2.0095	2.035
	$\pi^{-1}(k^h)^+$	2.0068	2.0357	2.046	2.0337	2.116
	$\pi^{-1}\Delta k^h$	0.0144	0.0709	0.00828	0.0242	0.081
3	$\pi^{-1}(k^h)^-$	2.990	2.9843	2.9751	2.9975	2.986
	$\pi^{-1}(k^h)^+$	3.0076	3.0156	3.0306	3.0245	3.101
	$\pi^{-1}\Delta k^h$	0.0176	0.0313	0.0555	0.027	0.115

TABLE 2.

Boundaries $(k^h)^-$ and $(k^h)^+$ and widths $\Delta k^h = (k^h)^+ - (k^h)^-$, of the first three stop bands for five values of R, obtained by numerical solution of (4.1) with $N + 1 = 100$

K/π		$R = 0.1$	$R = 0.2$	$R = 0.3$	$R = 0.4$	$R = 0.5$
1	$\pi^{-1}k^{-}_{(2)}$	0.9959	0.9698	0.9117	0.827	0.697
	$\pi^{-1}k^{+}_{(2)}$	1.0041	1.0298	1.0907	1.190	1.366
	$\pi^{-1}\Delta k_{(2)}$	0.0082	0.06	0.179	0.363	0.669
2	$\pi^{-1}k^{-}_{(2)}$	1.9927	1.9652	1.9637	2.0099	2.036
	$\pi^{-1}k^{+}_{(2)}$	2.0072	2.0361	2.0462	2.0341	2.116
	$\pi^{-1}\Delta k_{(2)}$	0.0145	0.0709	0.4092	0.0242	0.08
3	$\pi^{-1}k^{-}_{(2)}$	2.990	2.9855	2.976	2.9992	2.985
	$\pi^{-1}k^{+}_{(2)}$	3.0088	3.0166	3.0314	3.025	3.104
	$\pi^{-1}\Delta k_{(2)}$	0.0188	0.0311	0.0554	0.0258	0.119

TABLE 3.

Boundaries and widths of the first three stop bands calculated by second order perturbation theory, (3.28). Numerical integration was used for calculating the V_{m0}, and only terms with $m \leq 12$ were included.

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